参考：https://batapara.com/archives/differential-equation-total-derivative2.html/
次の完全微分型の微分方程式を解け。
【問題】

$$
\begin{aligned}
&(1)\quad y\,dx + x\,dy = 0\\
&(2)\quad 2x\,dx + 2y\,dy = 0\\
&(3)\quad -\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = 0\\
&(4)\quad \cos x\,dx - \sin y\,dy = 0\\
&(5)\quad e^x \cos y\,dx - e^x \sin y\,dy = 0\\
&(6)\quad (1+y)\,dx + (x+4y)\,dy = 0\\
&(7)\quad (6x+4y)\,dx + (4x+4y)\,dy = 0\\
&(8)\quad (3x^2+2x+3y^2)\,dx + (6xy-3y^2)\,dy = 0\\
&(9)\quad (3x^2+6xy-2y^2)\,dx + (3x^2-4xy+3y^2)\,dy = 0\\
&(10)\quad (3x^2y-2xy+1)\,dx + (x^3-x^2-4y^3)\,dy = 0\\
&(11)\quad y' = \frac{2x-6y}{6x-18y}\\
&(12)\quad y' = \frac{3x^2-2xy+y^2}{x^2-2xy-3y^2}\\
&(13)\quad (2x+e^y)\,dx + (xe^y+\sin y+y\cos y)\,dy = 0\\
&(14)\quad \left(\frac{1}{2}\sqrt{\frac{1}{xy}} - \frac{1}{2}\sqrt{\frac{y}{x^3}}\right)dx + \left(-\frac{1}{2}\sqrt{\frac{x}{y^3}} + \frac{1}{2}\sqrt{\frac{1}{xy}}\right)dy = 0\\
&(15)\quad \left(\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}x^{-\frac{2}{3}}y^{\frac{1}{3}}\right)dx + \left(\frac{2}{3}x^{\frac{1}{3}}y^{-\frac{2}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\right)dy = 0
\end{aligned}
$$

【解答】

(1) の解答：

$$\begin{aligned} & y\,dx + x\,dy = 0 \\ \Leftrightarrow\quad & d(xy) = 0 \\ \Leftrightarrow\quad & xy = C \quad \blacksquare \end{aligned}$$

(2) の解答：

$$\begin{aligned} & 2x\,dx + 2y\,dy = 0 \\ \Leftrightarrow\quad & d(x^2 + y^2) = 0 \\ \Leftrightarrow\quad & x^2 + y^2 = C \quad \blacksquare \end{aligned}$$

(3) の解答：

$$\begin{aligned} & -\frac{y}{x^2}\,dx + \frac{1}{x}\,dy = 0 \\ \Leftrightarrow\quad & d\left(\frac{y}{x}\right) = 0 \\ \Leftrightarrow\quad & \frac{y}{x} = C \quad \blacksquare \end{aligned}$$

(4) の解答：

$$\begin{aligned} & \cos x\,dx - \sin y\,dy = 0 \\ \Leftrightarrow\quad & d\left(\sin x + \cos y\right) = 0 \\ \Leftrightarrow\quad & \sin x + \cos y = C \quad \blacksquare \end{aligned}$$

(5) の解答：

$$\begin{aligned} & e^x \cos y\,dx - e^x \sin y\,dy = 0 \\ \Leftrightarrow\quad & d\left(e^x \cos y\right) = 0 \\ \Leftrightarrow\quad & e^x \cos y = C \quad \blacksquare \end{aligned}$$

(6) の解答：

$$\begin{aligned} & (1 + y)\,dx + (x + 4y)\,dy = 0 \\ \Leftrightarrow\quad & d\left(x + xy + 2y^2\right) = 0 \\ \Leftrightarrow\quad & x + xy + 2y^2 = C \quad \blacksquare \end{aligned}$$

(7) の解答：

$$\begin{aligned} & (6x + 4y)\,dx + (4x + 4y)\,dy = 0 \\ \Leftrightarrow\quad & d\left(3x^2 + 4xy + 2y^2\right) = 0 \\ \Leftrightarrow\quad & 3x^2 + 4xy + 2y^2 = C \quad \blacksquare \end{aligned}$$

(8) の解答：

$$\begin{aligned} & (3x^2 + 2x + 3y^2)\,dx + (6xy - 3y^2)\,dy = 0 \\ \Leftrightarrow\quad & d\left(6xy - 3y^2\right) = 0 \\ \Leftrightarrow\quad & 6xy - 3y^2 = C \quad \blacksquare \end{aligned}$$

(9) の解答：

$$\begin{aligned} & (3x^2 + 6xy - 2y^2)\,dx + (3x^2 - 4xy + 3y^2)\,dy = 0 \\ \Leftrightarrow\quad & d\left(x^3 + 3x^2 y - 2xy^2 + y^3\right) = 0 \\ \Leftrightarrow\quad & x^3 + 3x^2 y - 2xy^2 + y^3 = C \quad \blacksquare \end{aligned}$$

(10) の解答：

$$
\begin{aligned}
& (3x^2y - 2xy + 1)\,dx + (x^3 - x^2 - 4y^3)\,dy = 0 \\
\Leftrightarrow\quad & d\left(x^3y - x^2y + x - y^4\right) = 0 \\
\Leftrightarrow\quad & x^3y - x^2y + x - y^4 = C \quad \blacksquare
\end{aligned}
$$

(11) の解答：

$$
\begin{aligned}
& y' = \frac{2x - 6y}{6x - 18y} \\
\Leftrightarrow\quad & (2x - 6y)\,dx + (-6x + 18y)\,dy = 0 \\
\Leftrightarrow\quad & d\left(x^2 - 6xy + 9y^2\right) = 0 \\
\Leftrightarrow\quad & x^2 - 6xy + 9y^2 = C \quad \blacksquare
\end{aligned}
$$

(12) の解答：

$$
\begin{aligned}
& y' = \frac{3x^2 - 2xy + y^2}{x^2 - 2xy - 3y^2} \\
\Leftrightarrow\quad & (3x^2 - 2xy + y^2)\,dx + (-x^2 + 2xy + 3y^2)\,dy = 0 \\
\Leftrightarrow\quad & d\left(x^3 - x^2y + xy^2 + y^3\right) = 0 \\
\Leftrightarrow\quad & x^3 - x^2y + xy^2 + y^3 = C \quad \blacksquare
\end{aligned}
$$

(13) の解答：

$$
\begin{aligned}
& (2x + e^y)\,dx + (xe^y + \sin y + y\cos y)\,dy = 0 \\
\Leftrightarrow\quad & d\left(x^2 + xe^y + y\sin y\right) = 0 \\
\Leftrightarrow\quad & x^2 + xe^y + y\sin y = C \quad \blacksquare
\end{aligned}
$$

(14) の解答：

$$
\begin{aligned}
& \left(\frac{1}{2}\sqrt{\frac{1}{xy}} - \frac{1}{2}\sqrt{\frac{y}{x^3}}\right)\,dx + \left(-\frac{1}{2}\sqrt{\frac{x}{y^3}} + \frac{1}{2}\sqrt{\frac{1}{xy}}\right)\,dy = 0 \\
\Leftrightarrow\quad & d\left(\sqrt{\frac{y}{x}} + \sqrt{\frac{x}{y}}\right) = 0 \\
\Leftrightarrow\quad & \sqrt{\frac{y}{x}} + \sqrt{\frac{x}{y}} = C \quad \blacksquare
\end{aligned}
$$

(15) の解答：

$$
\begin{aligned}
& \left(\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}x^{-\frac{2}{3}}y^{\frac{1}{3}}\right)\,dx + \left(\frac{2}{3}x^{\frac{1}{3}}y^{-\frac{2}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\right)\,dy = 0 \\
\Leftrightarrow\quad & d\left(x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}\right) = 0 \\
\Leftrightarrow\quad & x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = C_1 \\
\Leftrightarrow\quad & \left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^2 = C_1 \\
\Leftrightarrow\quad & x^{\frac{1}{3}} + y^{\frac{1}{3}} = C \quad \blacksquare
\end{aligned}
$$